Grounding Ground*

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Abstract

The Problem of Iterated Ground is to explain what grounds truths about ground: if Γ grounds ϕ, what grounds that Γ grounds ϕ? This paper develops a novel solution to this problem. The basic idea is to connect ground to explanatory arguments. By developing a rigorous account of explanatory arguments we can equip operators for factive and non-factive ground with natural introduction and elimination rules. A satisfactory account of iterated ground falls directly out of the resulting logic: non-factive grounding claims, if true, are *zero-grounded* in the sense of Fine.

1 Introduction

If Γ's being the case grounds ϕ's being the case, what grounds *that* Γ's being the case grounds ϕ's being the case?[1] This is the *Problem of Iterated Ground*. (Dasgupta 2014c; Bennett 2011; and deRosset 2013) have grappled with this problem from the point of view of metaphysics. But iterated ground is a problem not just for metaphysicians: the existing logics of ground[2] have had nothing to say about such iterated grounding claims. In this paper I propose a novel account of iterated ground and develop a logic of iterated ground. The account—what I will call the Zero-Grounding Account (ZGA for short)—is based

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[1] Here Γ are some (true) propositions and ϕ is a (true) proposition. For the official formulation of claims of ground, see § 2 below. In the interest of readability I will not distinguish carefully between use and mention throughout.

[2] Fine 2012b; Correia 2010, 2014; Schnieder 2011; Poggiolesi forthcoming.

on three mutually supporting ideas: (i) taking non-factive ground as a primitive notion of ground; (ii) tying non-factive ground to explanatory arguments; and (iii) holding that true non-factive grounding claims are zero-grounded (in Fine's sense).

A notion of ground is *factive* if the truth of "Γ grounds ϕ" entails that each $\gamma \in \Gamma$ as well as ϕ is true; the notion is non-factive otherwise. Most authors take a factive notion of ground as their primitive; I adopt a non-factive notion as primitive. Taking a non-factive notion of ground as basic allows one to solve the Problem of Iterated Ground for factive ground: if Δ factively grounds ϕ then this is grounded in Δ's non-factively grounding ϕ together with Δ's being the case.[3] This, of course, just shifts the bump under the rug: what grounds that Δ non-factively grounds ϕ?

(Fine 2012a, pp. 47–48) distinguished between a truth's being *ungrounded*, on the one hand, and having the *empty ground* or being *zero-grounded* on the other. Crucially, being zero-grounded is a way of being *grounded*. I show that if Δ's non-factively grounding ϕ is zero-grounded we have a formally satisfactory solution to the Problem of Iterated Ground. To go beyond a merely formal solution we must answer two questions: (i) What does it mean to say that a truth is zero-grounded? (ii) Why should we believe that (true) non-factive grounding claims are zero-grounded?

We answer these questions by tying non-factive ground to *explanation*. The basic idea is that for Δ to non-factively ground ϕ just is for there to be a special type of argument from premisses (exactly) Δ to conclusion ϕ—what we can call a *metaphysically explanatory argument*. If one accepts this connection between ground and metaphysically explanatory arguments, the notion of zero-grounding is unproblematic: a truth is zero-grounded if it is the conclusion of an explanatory argument from the empty collection of premisses. The seemingly mysterious distinction between being *ungrounded* and being *zero-grounded* is a special case of the familiar distinction between not being derivable and being derivable from the empty collection of premisses.

In response to the second question, I do not simply postulate that non-factive grounding claims are zero-grounded. If the claim that Δ non-factively grounds ϕ just is the claim that there exists an explanatory argument from Δ to ϕ there are compelling reasons for holding that the claim that Δ non-factively grounds ϕ—if true—is zero-grounded. To substantiate this I show how to develop a logic of iterated ground—the Pure Logic of Iterated Strict Full Ground (PLISFG). A novel feature of PLISFG is that its deductive system distinguishes between explanatory arguments and what we may call "plain" arguments. This allows us to equip factive and non-factive grounding operators with natural introduction and elimination rules. (In fact, the rules are proof-theoretically harmonious.) Together these rules entail that true non-factive grounding claims are zero-grounded.

[3] It is not strictly speaking necessary to hold that true factive grounding claims are partially grounded in non-factive grounding claims (§ 10). However, assuming this allows for a smoother presentation; and, as we will see, it does no harm.

Overview of the paper

§ 2 explains how the various notions of ground are to be understood. § 3 rehearses a serious problem posed by claims of iterated ground. § 4 formally states the ZGA. § 5 sketches a graph-theoretic account of ground, discusses how the graphs are to be understood, and shows how the zero-grounding of non-factive grounding claims is a natural consequence. § 6 show how we can understand ground in terms of explanatory arguments and develops a deductive system distinguishing between explanatory and merely plain arguments. § 7 shows how to find introduction rules for the grounding operators; § 8 uses an inversion principle to find matching elimination rules; these rules have the consequence that non-factive grounding claims, if true, are zero-grounded. § 9 defends the ZGA against the objection that every true non-factive grounding claim has the same (empty) ground. § 10 compares the ZGA with the "Straightforward Account" (SFA) offered by (Bennett 2011; and deRosset 2013) and argues that even the SFA needs zero-grounding. The paper concludes with some issues for further work in § 11. There are two technical appendices. Appendix A states introduction and elimination rules for the Pure Logic of Iterated Strict Full Ground (PLISFG) in an "amalgamation-friendly" form to facilitate comparison with Fine's Pure Logic of Ground (PLG). Appendix B develops a graph-theoretic semantics and uses it to show that PLISFG is a conservative extension of a subsystem of Fine's Pure Logic of Ground.[4]

2 Ground and Explanation

I take ground to be an explanatory notion. As I will understand ground to say that $\phi_0, \phi_1, \ldots$ ground ϕ just is to say that $\phi_0, \phi_1, \ldots$ explain ϕ in a distinctively metaphysical way.[5] The explanatory connection between the grounds and what they ground is very intimate; following Fine I take the grounds to explain the grounded in the sense "that there is no stricter or fuller account of that in virtue of which the explanandum holds. If there is a gap between the grounds and what is grounded, then it is not an explanatory gap." (Fine 2012a, p. 39)[6] Here are some plausible cases of ground:

[4]The graph-theoretic semantics can be extended to a semantics for all the grounding operators of Fine's Pure Logic of Ground—and more besides. It is also possible to find introduction and elimination rules for these operators—and more besides. This is a task for another occasion.

[5] I should flag a controversy here. That grounding is intimately connected with explanation is widely accepted; that Γ's non-factively grounding ϕ just consists in Γ's explaining ϕ in a distinctive way is *not* uncontroversial, though it is accepted by (Fine 2001, 2012a; and Dasgupta 2014a,b). An alternative view would take grounding to be a (the?) distinctive relation of determination that "underwrites" such metaphysical explanations. On this view grounding stands to metaphysical explanation as causation stands to causal explanation. (For such a view see (Audi 2012b, p. 688; and Audi 2012a) and (Schaffer 2012, forthcoming).) I will not attempt to refute this position here. Many of the made in this paper can, in any case, be appropriated by the defenders of this other view.

[6](Fine 2012a, pp. 38–40) distinguishes between metaphysical, normative and natural ground. Here we will only discuss metaphysical ground: it is only with metaphysical ground that the connection between the grounds and that which they ground is this intimate.

(1) A conjunction is grounded in its conjuncts.

(2) A disjunction is grounded in its true disjuncts.[7]

(3) *a*'s being G is grounded in *a*'s being F, where F is a determinate of the determinable G.

Several notions of ground have been distinguished in the literature. The one that has been the focus of the debate over iterated ground is *factive*, *full*, *mediate*, *strict* ground. Adopting the notation of (Fine 2012a,b) we express claims of ground using a sentential operator "$<$".[8]

"$<$" has variable arity on the left: if Δ is any set[9] of sentences and ϕ is a sentence then the result of concatenating the sentences in Δ (in any order) with $<$ and ϕ is a sentence.[10] Since nobody thinks that the order of the sentences in Δ matters to whether Δ grounds ϕ, we disregard order and (ambiguously) write $\Delta < \phi$ for the resulting sentence. Δ are here the *grounds* while ϕ is the *grounded*. We can pronounce a grounding claim $\Delta < \phi$ as "ϕ because Δ". Note that we allow both infinite and empty Δ; the latter means that $<\phi$ is well-formed. Since the sentential operator locution can be cumbersome we often nominalize and, unofficially, speak of grounding as a *relation* between *truths*.

The following logical features are commonly taken to hold of $<$. We will take them as adequacy constraints on a logic of the operator $<$: if a proposed logic of ground fails to validate these features we will not accept it.

$<$ is *factive* in the sense that if $\Delta < \phi$, then ϕ and each $\delta \in \Delta$ is true. It is *full* in the sense that if $\Delta < \phi$ is the case then nothing need be added to Δ in order to explain why it is the case that ϕ: its being the case that Δ fully accounts for its being the case that ϕ. It is *mediate* in the sense that we allow Δ to ground ϕ by way of grounding some ψ that also grounds ϕ.[11] It is *strict* in the sense

[7] Though this is problematic because of the paradoxes of ground discussed in (Fine 2010a). We set the paradoxes of ground aside for the purposes of this paper.

[8] I here follow (Fine 2001, 2012a,b; Schnieder 2011; Dasgupta 2014c; and Correia 2010) in expressing ground by means of sentential operators. The alternatives discussed in the literature are (i) to treat grounding as a relation between facts; (Rosen 2010; Audi 2012a,b; deRosset 2013; Bennett 2011; and Trogdon 2013a); and (ii) to treat grounding as a relation that can hold between objects in arbitrary ontological categories (Schaffer 2009). The reasons for expressing ground by means of a sentential operator are frankly pragmatic. First, by expressing ground using sentential operators one can remain neutral on some vexed issues concerning the existence and nature of facts (cf. Fine 2001, 2012a,b; and Correia 2010). Secondly, it is easier to formulate a logic of ground if one expresses ground using sentential operators. Thirdly, the crucial notion of zero-grounding is also easier to make sense of if we express claims of ground using sentential operators. That being said, it would be possible to reformulate much of what follows if one favored treating grounding as a relation between facts.

[9] It might be better to let Δ be a *multiset*. (See footnote 54.) In the main text I have opted for simplicity.

[10] This is a *many-one* notion of ground. One might want to allow $<$ to have variable arity on the right as well: when Γ and Δ are any two *sets* the result of concatenating the sentences in Γ with $<$ and Δ is a grounding claim. (Dasgupta 2014b) argues that there is a *non-distributive* notion of many-many ground. $\Gamma < \Delta$ can be the case without there being an I such that $\Gamma = \bigcup_{i \in I} \Gamma_i$, $\Delta = \bigcup_{i \in I} \{\delta_i\}$ and $\Gamma_i < \delta_i$ for each $i \in I$. (Litland forthcoming[b]) extends Fine's truthmaker semantics to develop a logic of many-many ground. While such a notion of ground is of considerable interest we will not discuss it further in this paper—though see footnote 35.

[11] More generally: we allow that Δ grounds ϕ by way of there being a decomposition

that if $\Delta < \phi$ then it is impossible for each $\delta \in \Delta$ to be the case while ϕ helps explain a $\delta' \in \Delta$. In the special case of one-one grounding claims—i.e., claims of the form $\phi < \psi$—this has the consequence that grounding is *irreflexive*, *transitive* and *asymmetric*.[12]

The claims of ground in (1), (2), and (3) above are all plausible cases of factive, strict, full, mediate ground.

Central to the ZGA is a notion of *non-factive* (full, mediate, strict) ground. Whereas only truths can factively ground, even falsehoods can non-factively ground. We use "$\Rightarrow$" as a sentential operator for this notion; it has the same grammar as $<$. Much more will be said about $\Rightarrow$ later; for now it suffices to know that if (each γ in) Γ is the case and $\Gamma \Rightarrow \phi$ is the case then $\Gamma < \phi$.[13] The claims (1), (2), and (3) are also true non-factive grounding claims.

3 A Status Problem

While claims of iterated ground are interesting in their own right they also give rise to serious problems. Since problems of this sort are now well-known, we consider only a simple case.[14] (For ease of expression we talk as if grounding was a relation between truths. Note that the official formulation would require quantification into "sequence-of-sentences" position.)

Consider first the following principle:

(Foundation) There are some truths Δ such that each $\delta \in \Delta$ is ungrounded and such that each truth ϕ is either in Δ or else is grounded in some $\Delta_0 \subseteq \Delta$. Let us call such a collection of truths a *foundation*.

Many philosophical disputes can be construed as disputes about whether there could be foundations consisting only of certain types of truths. For instance, a physicalist might think that there is a foundation consisting only of physical truths: no mental truth need be entered into the foundation. A metaethical naturalist might think that there is a foundation consisting only of non-normative truths: no normative truth need be entered into the foundation.

A physicalist might want to go further. Not only is there no need for mental *truths* in the foundation; distinctively mental *objects* and *properties* are not

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$\Delta_0, \Delta_1, \ldots, \Delta_\omega$ of Δ and some $\psi_0, \psi_1, \ldots$ such that Δ_i grounds ψ_i for each i and $\psi_0, \psi_1, \ldots, \Delta_\omega$ taken together ground ϕ.

[12] I should note that these principles have been contested. (Jenkins 2011; Wilson 2014; Correia 2014, § 7.3; and Krämer 2013) doubt that grounding is irreflexive; (Dasgupta 2014c) doubts that grounding is asymmetric; and (Schaffer 2012), *contra* (Schaffer 2009, pp. 375–6), argues that grounding is not transitive. (For a defense of the principles see Raven 2013; and Litland 2013).

[13] For more on the distinction between factive and non-factive ground, see (Fine 2012a, pp. 48–50)

[14] Essentially this problem is discussed by (Bennett 2011; deRosset 2013; Dasgupta 2014c) The problem does not essentially turn on every truth being grounded in some *un*grounded truths. Problems like this arise even if one allows infinitely descending chains of ground. (Sider 2012, pp. 143–145), e.g., formulates a Status Problem turning on Fine's notions of a fact's being *constitutive of reality* and its being *factual* that p (for these notions, see Fine 2001, 2010b).

required either:[15] there is a foundation $\mathcal{F}$ such that no mental objects and properties are constituents of the truths of $\mathcal{F}$.

What a physicalist might want to say is that no mental object is O-fundamental in the following sense:

(Object-Fundamentality) An object a is *O-fundamental* iff the object a figures in an ungrounded truth

If that is right, the physicalist is committed to:

(Derivative Objects) There is an object a that is not O-fundamental.

The problem is that if truths about ground are themselves ungrounded then *every* object is O-fundamental. For consider some object a: either there is nothing that grounds the truth that a exists or there is something that grounds this truth. If the former, then a is O-fundamental since the truth that a exists is fundamental. If the latter, suppose that the truth that a exists is grounded in the truths Γ. Then this truth, *viz.*, the truth that a's existence is grounded in Γ, is a further truth. If this truth is ungrounded then we again get that a is O-fundamental since the truth that the existence of a is grounded in Γ will be in every foundation.

We will take it as a constraint on an account of iterated ground that it not commit us to every object's being O-fundamental. There are two reasons for imposing the constraint. First, the question whether every object is O-fundamental seems like a substantive one: it should not be settled in the above trivial manner (cf. Dasgupta 2014c). Second, whether or not it is a substantive matter whether every object is O-fundamental the other views on iterated ground do avoid the conclusion that all objects are O-fundamental. For dialectical purposes the ZGA, too, had better not have this consequence.

4 The Zero-Grounding Account

Recall that when Δ is any number of sentences and ϕ is a sentence, then $\Delta{<}\phi$ and $\Delta \Rightarrow \phi$ are sentences; in particular, if Δ is $\emptyset$, the empty collection of sentences, then ${<}\phi$ and $\Rightarrow \phi$ are sentences. If ${<}\phi$, then ϕ is grounded: there is a collection Δ, *viz.*, the empty collection, of sentences such that $\Delta < \phi$ is true. Let us, for now, take the notion of zero-grounding for granted and see how it allows us formally to solve the problem of Iterated Ground.

The ZGA holds that $\Delta < \phi$ is grounded in Δ together with $\Delta \Rightarrow \phi$ and that $\Delta \Rightarrow \phi$ is strictly fully zero-grounded. That is, it is the case that $\emptyset < (\Delta \Rightarrow \phi)$. We avoid the conclusion that every object is fundamental. If $\Delta < \phi$ is the case then this is grounded in Δ together with $\Delta \Rightarrow \phi$. The latter is zero-grounded and hence grounded. If the claim ϕ concerns the object a, a will not be O-fundamental just on account of occurring in $\Delta \Rightarrow \phi$.

[15] See (deRosset 2013, pp. 3–6) for some reasons for thinking this.

What about the claim that $\Delta \Rightarrow \phi$ is zero-grounded? If this claim is ungrounded, the problem is only pushed back. The solution is obvious: if a truth ψ is zero-grounded then the truth that ψ is zero-grounded is itself zero-grounded.

We then get the following sequence of grounding claims.

$$\Delta < \phi \qquad (\Delta, \Delta \Rightarrow \phi) < (\Delta < \phi)$$

$$[(\Delta, \Delta \Rightarrow \phi), ((\Delta, \Delta \Rightarrow \phi) \Rightarrow (\Delta < \phi))] < ((\Delta, \Delta \Rightarrow \phi) < (\Delta < \phi))\ldots$$

Since $\emptyset < (\Delta \Rightarrow \phi)$ we simplify and get:[16]

$$\Delta < \phi \qquad \Delta < (\Delta < \phi) \qquad \Delta < (\Delta < (\Delta < \phi)) \quad \ldots$$

The ZGA gives rise to infinitely many grounding claims involving ϕ. We do not, however, have an infinitely descending chain of ground with ϕ on top: ϕ is grounded in Δ, but since ϕ is *not* grounded in $\Delta < \phi$ there is no regress.[17]

There is a superficial similarity with the accounts of (deRosset 2013) and (Bennett 2011). Their view is that if $\Delta < \phi$ then what grounds this is just Δ itself.[18] Let us call this the *Straightforward Account* (SFA). We will discuss the SFA in greater detail in § 10; here just note that the ZGA and the SFA differ even though they agree that $\Delta < \phi$ only if $\Delta < (\Delta < \phi)$. The ZGA holds that $\Delta < \phi$ is partially grounded in $\Delta \Rightarrow \phi$; the SFA denies this.[19]

To move beyond the merely formal we must answer the following two questions.

(4) What exactly does it mean to say that something is zero-grounded?

(5) How can we make it intelligible that truths of the form $\Delta \Rightarrow \phi$ are zero-grounded?

The question in (5) is not a request for the *grounds* for something's being zero-grounded: it is part of the view that if ϕ is zero-grounded, then the truth that ϕ is zero-grounded is itself zero-grounded (and so on). Rather, one is asking for a story making it *comprehensible* that something is zero-grounded. It is true that by taking non-factive grounding claims to be zero-grounded (if true) we solve the Status Problem. While this provides *some* reason to postulate that they are zero-grounded (if true), it does not help us understand why they should be zero-grounded. The key to answering these questions lies in answering the following one:

[16] I assume the Cut principle: if $\Gamma < \phi$ and $\Delta, \phi < \psi$, then $\Gamma, \Delta < \phi$. (The relevant case is $\Gamma = \emptyset$.)

[17] For more discussion of why there is no problem see (Bennett 2011, pp. 33–35; deRosset 2013, pp. 19–20; Rabin and Rabern 2015) In any case, if there were a problematic regress here a similarly problematic regress would arise for the other accounts of Iterated Ground: the ZGA is no worse off.

[18] A similar view is tentatively suggested in (Raven 2009).

[19] If we avail ourselves of the notion of *immediate* strict full ground we can state the difference perspicuously. (For the distinction between mediate and immediate ground, see (Fine 2012a, pp. 50–51).) According to the ZGA, $\Delta < \phi$ is *not* immediately strictly fully grounded in Δ—the immediate grounds for $\Delta < \phi$ are $\Delta, (\Delta \Rightarrow \phi)$ taken together; $\Delta < \phi$ is only mediatedly strictly fully grounded in Δ. According to the SFA, on the other hand, $\Delta < \phi$ would be immediately strictly fully grounded in Δ. Bennett and deRosset admittedly do not discuss the problem in terms of immediate strict full ground. But it seems clear that the natural way of developing their view is by insisting that Δ is the immediate full ground for $\Delta < \phi$.

(6) How should the notion of non-factive ground be understood?

We want to end up in the following situation. Consider conjunction: there is no mystery about why a true conjunction $\phi \wedge \psi$ is fully grounded in its conjuncts ϕ, ψ taken together. Once one understands what conjunction is one understands that a (true) conjunction is grounded in its conjuncts. Similarly, once we get clear on what non-factive grounding is there will be no mystery why a true claim of non-factive ground is zero-grounded.

5 Ground, Machines, and Graphs

To see both what *zero*-grounding is and why non-factive grounding claims would be zero-grounded (if true) it is useful to begin with a picture of *grounding*.

Think of a machine generating truths from other truths.[20] The machine is fed truths, churning out truths grounded in the truths it is fed. A truth is *ungrounded* if the machine never churns it out; a truth is *zero-grounded* if the machine churns it out when it is fed *no* input.

In terms of this picture, why would the machine give the verdict that $\Delta \Rightarrow \phi$ is zero-grounded if true? Think of it like this. When the machine is fed no input the machine, instead of remaining idle, "simulates" the results of being fed various input. In simulating what happens when it is fed the propositions Δ the machine proceeds just as it would have if it in fact had been fed Δ as input. If, when running the simulation, the machine churns out ϕ, the machine ends the simulation and churns out $\Delta \Rightarrow \phi$. Since the machine was fed no input this means that $\Delta \Rightarrow \phi$ is zero-grounded if true.

The machine picture is closely related to a graph-theoretical picture often employed in discussions of ground. Let us develop this picture both for non-factive and factive ground.[21] A *directed hypergraph* is a tuple $\mathcal{G} = \langle V, \mathcal{A}, t, h\rangle$. Here V is a collection of *vertices*—think of these as propositions. $\mathcal{A}$ is a collection of hyperarcs. t, h are functions $t, h : \mathcal{A} \to \mathcal{P}(V)$. If $A \in \mathcal{A}$, $t(A)$ is the *tail* of A and $h(A)$ is the *head* of A. We demand that $h(A)$ is a singleton.[22] Assume that the vertices are propositions; we can then speak of an arc A as being *from* the propositions $\phi_0, \phi_1, \ldots$ *to* the proposition ϕ. In terms of the machine picture, an arc $A \in \mathcal{A}$ corresponds to the application of a mechanism inside the machine. $t(A)$ are the propositions the mechanism operates on; $h(A)$ is the result of the mechanism's operating on $t(A)$.[23] For simplicity, we assume that $\mathcal{A}$ is *chained*: if $A_0, A_1, \ldots, B$ are arcs such that $t(B) = \{v_0, v_1, \ldots, w_0, w_1, \ldots\}$ and $h(A_0) = v_0, h(A_1) = v_1, \ldots$, then there is an arc C such that $t(C) = t(A_0) \cup t(A_1) \cup \cdots \cup \{w_0, w_1, \ldots\}$ and $h(C) = h(B)$.[24]

[20] The analogy is from (Fine 2012a, pp. 47–48). The extension to non-factive ground is new here.

[21] (Schaffer 2009) and (deRosset forthcoming) have also used graph-theoretic ideas in connection with grounding. (See also (Schaffer forthcoming) for a related approach.)

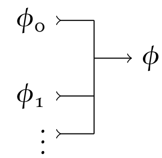
[22] To capture a many-many notion of ground we would lift this restriction.

[23] Note that A represents not the mechanism itself but rather an application of the mechanism. We will see why this might matter in § 9.

[24] Without this assumption the graphs would capture immediate and not mediate ground. In

The following two graphical representation of a hyperarc with tail $\{\phi_0, \phi_1, \dots\}$ and head ϕ are usefully kept in mind.

$$\frac{\phi_0 \quad \phi_1 \quad \dots}{\phi}$$



We can now say that some propositions $\phi_0, \phi_1, \dots$ *non-factively ground* a proposition ϕ iff there is an arc A such that $h(A)$ is $\{\phi\}$ and $t(A)$ is $\{\phi_0, \phi_1, \dots\}$. A proposition ϕ is (non-factively) *zero-grounded* if there is an arc A with the empty tail the head of which contains the proposition ϕ. A proposition ϕ is *ungrounded* if there is no arc A with $h(A) = \{\phi\}$.

To deal with factive ground we need the notion of a *pointed* directed hypergraph. This is a tuple $\langle V, F, \mathcal{A}, h, t \rangle$. Here $\langle V, \mathcal{A}, h, t \rangle$ is a directed hypergraph and $F \subseteq V$ is a set of vertices closed under taking heads; that is, if $F_0 \subseteq F$ and $A \in \mathcal{A}$ is such that $t(A) = F_0$ then $h(A) \subseteq F$.[25] Think of F as the actually obtaining propositions—the facts. We now say that $\phi_0, \phi_1, \dots$ *factively* ground ϕ if $\phi_0, \phi_1, \dots \subseteq F$ and $\phi_0, \phi_1, \dots$ non-factively ground ϕ. Since F is closed under taking heads every truth that is non-factively zero-grounded is in F.

Graph theory gives us nice formal models of ground, but how does it help explain why truths of the form $\Delta \Rightarrow \phi$ are zero-grounded?

Consider an arc A from some propositions $\phi_0, \phi_1, \dots$ to a proposition ϕ. What does this arc represent? One possibility is to take A to represent the *truth* that $\phi_0, \phi_1, \dots$ non-factively ground ϕ. I would like to defend a different view. What represents that $\phi_0, \phi_1, \dots$ ground ϕ is not an arc with tail $\{\phi_0, \phi_1, \dots\}$ and head ϕ, but rather *that* there is such an arc. What, then, do the arcs represent?

If we adopt the machine picture we can say that an arc from $\phi_0, \phi_1, \dots$ to ϕ represents an application of a mechanism to the propositions $\phi_0, \phi_1, \dots$, an application that yields the proposition ϕ. (And so: what represents that $\phi_0, \phi_1, \dots$ ground ϕ is not the application of a mechanism taking us from $\phi_0, \phi_1, \dots$ to ϕ but rather the fact *that* there is an application of a mechanism taking us from input $\phi_0, \phi_1, \dots$ to output ϕ.)

If one thinks of the arcs as applications of mechanisms the arcs are not propositional; and it does not make sense to ask what grounds them. However, even if the application of a mechanism taking us from input $\phi_0, \phi_1, \dots$ to output ϕ is not propositional—and so not apt to be grounded—there is still the proposition *that* there is such an application of a mechanism. The question what grounds this proposition is legitimate. If we think of the proposition $\phi_0, \phi_1, \dots \Rightarrow \phi$ as the proposition that there is an application of a mechanism

many ways it would be better not to assume that $\mathcal{A}$ is chained; but this raises technical problems that are best set aside here.

[25] Some further conditions need to be imposed, but we do not have to go into this now. See appendix B.

with input $\phi_0, \phi_1, \ldots$ and output ϕ the above story about simulation suggests why (true) propositions of the form $\phi_0, \phi_1, \ldots \Rightarrow \phi$ should be zero-grounded.

Suggestive though the machine picture is, it is not to be taken literally. We should provide a less pictorial account of what the arcs represent and what claim is being made by a non-factive grounding claim $\phi_0, \phi_1, \ldots \Rightarrow \phi$.

6 Ground and Explanatory Arguments

Above I wrote that we may think of an arc between some propositions $\phi_0, \phi_1, \ldots$ and a proposition ϕ as the application of a mechanism to the propositions $\phi_0, \phi_1, \ldots$; in the introduction I wrote that I would tie ground closely to explanatory arguments. These pictures are related. Think of an arc between $\phi_0, \phi_1, \ldots$ and ϕ as an *explanatory inference* from $\phi_0, \phi_1, \ldots$ to ϕ. We may represent this graphically as follows: $\dfrac{\phi_0 \quad \phi_1}{\phi}$. More generally, think of a hyperarc A as an *explanatory argument from* the propositions in the tail of A *to* the proposition in the head of A.[26] One can now think of the machine as encoding the explanatory arguments, with different explanatory arguments corresponding to different mechanisms.

We can now say that $\phi_0, \phi_1, \ldots$ non-factively ground ϕ if there is an explanatory argument from $\phi_0, \phi_1, \ldots$ to ϕ. And we can say that $\phi_0, \phi_1, \ldots$ factively ground ϕ if there is an explanatory argument from $\phi_0, \phi_1, \ldots$ to ϕ and $\phi_0, \phi_1, \ldots$ are the case.[27]

A main contribution of this paper is showing how we can develop a mathematically rigorous theory of metaphysically explanatory arguments; having done this a satisfactory logic of iterated ground drops out naturally.

I will not attempt to give a non-circular account of what makes something an explanatory argument: the notion is taken as a primitive. That is not to say that the notion cannot be elucidated. The intuitive idea is that the explanatory arguments are composed from basic explanatory *inferences*. Plausible cases of explanatory inference are conjunction-introduction, disjunction-introduction and the inference from a is F to a is G—where F is a determinate of the determinable G. (These correspond to the uncontroversial cases of ground mentioned in § 2 above.) More generally, whenever one thinks that Γ immediately grounds ϕ, the ZGA holds that the inference from Γ to ϕ is explanatory. Flippantly put, one obtains basic explanatory inferences as follows. Take a claim of strict full immediate ground—$\Gamma < \phi$ for instance—rotate it 90° clockwise and replace $<$ with a horizontal line: the result is an explanatory inference $\dfrac{\Gamma}{\phi}$.

It is appropriate for a *logic* of iterated ground to remain silent on which are the basic explanatory inferences: which particular inferences are explanatory

[26] It is in order to sustain this interpretation that we require that the hyperarcs be closed under *Cut*.

[27] (deRosset 2013, pp. 12-13) comes quite close to this idea, but he focuses on factive grounding and is not (there) trying to develop a logic of ground.

is (by and large) a material and not a formal matter and so one on which logic remains silent. That being said, for the purpose of illustration, let us assume that conjunction and disjunction introduction are explanatory inferences in this sense. If one thinks that a conjunction is not grounded in the conjuncts and a disjunction is not always grounded in the true disjuncts one should substitute some inferences that are obtained from the one's favorite claims of (immediate) ground in the way indicated above.

What logic can do is tell us how explanatory inferences interact and how they are chained together to form explanatory arguments. To do this we need a deductive system that distinguishes between two types of argument. This is the first technical innovation of this paper. Let us use uppercase calligraphic letters $\mathcal{E}, \mathcal{D}, \mathcal{F}, \ldots$ (possibly with subscripts) as variables over arguments. One type of argument is the *explanatory* argument: if there is an explanatory argument $\mathcal{E}$ from premisses Δ to conclusion ϕ, then if Δ is the case its being the case that Δ fully explains its being the case that ϕ. We also have the plain arguments: if there is a plain argument $\mathcal{E}$ from Δ to ϕ, then if Δ is true, ϕ, too, is true. With a plain argument, however, there is no guarantee that Δ, if true, explains ϕ.[28] To indicate that an argument $\mathcal{E}$ is explanatory we may write $\mathcal{E}(e)$; similarly, we may write $\mathcal{E}(p)$ to indicate that the argument $\mathcal{E}$ is merely plain.

Let us be more rigorous about what exactly arguments are. An argument is a quadruple $\mathcal{T} = \langle T, \leq, L, D \rangle$ where $\langle T, \leq, L \rangle$ is a labeled rooted tree. (T is the set of nodes, $\leq$ is the tree-order and $L\colon T \to P$ is a function assigning labels to the nodes of T. We may think of P as the class of propositions.) We demand that there is $T_0 \subseteq T$ such that each $t \in T_0$ is a top node and such that for each $s \in T$ there is $t \in T_0$ such that $s \leq t$.[29] D is a function $T \to \mathcal{P}(T)$. For each $s \in T$, D assigns a set $D(s) \subseteq \{t \geq s | t \text{ is a top node}\}$. This "discharge function" allows us to keep track of which assumptions a line in the argument depends on. (This is required since we will consider arguments that discharge assumptions.) The *root* of the (sub)tree is the conclusion of the (sub)argument.[30] If s is the root of a (sub)tree T, then the propositions labeling the nodes in $D(s)$ is the set of (undischarged) premisses of T. If ϕ labels node s we often abuse notation and write $D(\phi)$ for $D(s)$.

When we write

$$\begin{array}{c} \mathcal{E} \\ \phi \end{array}$$

we mean that $\mathcal{E}$ is an argument with conclusion ϕ. It is often important to note what the premisses of an argument $\mathcal{E}$ are. The notation

[28] I should note that there is nothing in the formalism forcing plain (or even strict) arguments to be, say, classically valid arguments. This is important: one should not rule out that Δ grounds ϕ, even though Δ does not logically entail ϕ.

[29] Because of examples of the sort given in (Dixon forthcoming), (Rabin and Rabern 2015), and (Litland forthcoming[a]) we cannot require that there are no infinite branches in T.

[30] Rather: the proposition labeling the root of the tree is the conclusion of the argument. We will be sloppy about this since there should be no cause for confusion.

$$\begin{array}{c}\Gamma\\ \mathcal{E}\\ \phi\end{array}$$

is to mean that $\mathcal{E}$ is an argument with conclusion ϕ where each occurrence of a proposition $\gamma \in \Gamma$ labels some node in $D(\phi)$. (We do not demand that each node in $D(\phi)$ is labeled with a proposition in Γ.)

Rules of inference that involve discharge will be written as follows.

$$\dfrac{\dfrac{}{\phi_0, \phi_1, \phi_2, \ldots}\, 0,1,2,\ldots \\ \mathcal{E} \\ \phi}{\psi}\, 0,1,2,\ldots$$

This is to be understood as follows. $\mathcal{E}$ is an argument with conclusion ϕ where $D(\phi)$ are labeled with (amongst others) the propositions $\phi_0, \phi_1, \ldots$. In passing to the conclusion ψ we can discharge any node labeled with one of the ϕ_i. More precisely, if $S \subseteq D(\phi)$ is any set of nodes with labels only from $\phi_0, \phi_1, \ldots$, any argument of the above form where $D(\psi) = D(\phi) \setminus S$ is an application of the rule.

(This means that unless otherwise specified we allow both *vacuous* and *multiple* discharge: if a rule allows the discharge of premisses of a certain form we can discharge any number of premisses of that form.)

When $\Gamma = \gamma_0, \gamma_1, \ldots$ we often use the following more compact notation

$$\dfrac{\dfrac{}{\Gamma}\, 1 \\ \mathcal{E} \\ \phi}{\psi}\, 1$$

to indicate that we can discharge any of the $\gamma \in \Gamma$.

The principles governing how explanatory and plain arguments interact are depicted in figure 1. Officially, figure 1 is understood as follows. Assume given a collection of basic explanatory and plain arguments $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$. (One may think of these as collections of explanatory and plain *inferences*.) The explanatory and plain arguments over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ is the least $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ such that $\mathfrak{E}_e \subseteq \mathfrak{E}'_e$, $\mathfrak{E}_p \subseteq \mathfrak{E}'_p$ where $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ is closed under the constraints in figure 1. Some explanation of these principles are in order.

(Inclusion) is straightforward. If its being the case that Γ would explain why ϕ is the case, then it is certainly true that if Γ is the case then ϕ is the case. (Assumption) is standard natural deduction: we can write down any assumption we like, with the result being a plain argument. (Plain Chaining) is also unproblematic; it just tells us that we can chain together plain arguments to get plain arguments.

Inclusion Any explanatory argument is a plain argument.

Assumption For any ϕ, ϕ is a plain argument from ϕ to ϕ

Non-circularity If $\mathcal{E}$ is an explanatory argument from premisses $\phi, \delta_0, \delta_1, \ldots$ to ϕ, and $\mathcal{D}$ is a plain argument from Γ to ϕ and $\mathcal{D}_i$ is a plain argument to δ_i for each i, then for any ψ the following is a plain argument from $\Gamma, \Delta_0, \Delta_1, \ldots$ to ψ:

$$\dfrac{\begin{array}{cc} \Gamma & \ldots\Delta_i\ldots \\ \mathcal{D} & \ldots\mathcal{D}_i\ldots \\ \phi & \ldots\delta_i\ldots \end{array} \\ \mathcal{E} \\ \phi}{\psi}$$

Plain Chaining If $\mathcal{E}_i$ is a plain argument from Δ_i to ϕ_i for each i, and $\mathcal{D}$ is a plain argument to ϕ from $\phi_0, \phi_1, \ldots, \Gamma$ then

$$\begin{array}{c} \begin{array}{cccc} \Delta_0 & \Delta_1 & & \\ \mathcal{E}_0 & \mathcal{E}_1 & & \\ \phi_0 & \phi_1 & \ldots & \Gamma \end{array} \\ \mathcal{D} \\ \phi \end{array}$$

is a plain argument from $\Delta_0, \Delta_1, \ldots, \Gamma$ to ϕ.

Chaining If $\mathcal{E}_i$ is an explanatory argument from Δ_i to ϕ_i for each i, and $\mathcal{D}$ is an explanatory argument to ϕ from $\phi_0, \phi_1, \ldots, \Gamma$ then

$$\begin{array}{c} \begin{array}{cccc} \Delta_0 & \Delta_1 & & \\ \mathcal{E}_0 & \mathcal{E}_1 & & \\ \phi_0 & \phi_1 & \ldots & \Gamma \end{array} \\ \mathcal{D} \\ \phi \end{array}$$

is an explanatory argument from $\Delta_0, \Delta_1, \ldots, \Gamma$ to ϕ.

Figure 1: Arguments explanatory and plain

(Chaining) tells us that the result of chaining together explanations is itself an explanation. One might be concerned about this if one thought that explanation is not transitive. Whatever one thinks about that in general there should be no problem here: explanatory arguments correspond to *full* ground. If Γ grounds ϕ then Γ provides a *full* account of why it is the case that ϕ. It is hard to see how there could be counterexamples to transitivity for this notion of ground.[31] In any case, since we want to capture *mediate* ground we can always insist on closure under (Chaining).

(Non-circularity) requires further explanation. (We take take up some more philosophical issues in §§ 6.1 and 6.2.) The goal is to have explanatory arguments correspond to strict ground in the sense that $\phi_0, \phi_1, \ldots < \phi$ iff each of the ϕ_i is the case and there is an explanatory argument from $\phi_0, \phi_1, \ldots$ to ϕ. Since (partial strict) ground is irreflexive it cannot be possible for some $\phi, \psi_0, \psi_1, \ldots$ to be the case and for there to be an explanatory argument from $\phi, \psi_0, \psi_1, \ldots$ to ϕ. It will not do simply to say that there are no explanatory arguments from ϕ, Γ to ϕ, for any Γ. The problem is that this tells us nothing about what happens under the *supposition* that ϕ (partly) strictly grounds itself. And we need to use the assumption that ϕ contributes to grounding itself in subordinate arguments. (Non-circularity) gets around this problem by expressing the irreflexivity of ground as a closure-condition on the class of strict and plain arguments. If, *per impossibile*, ϕ did contribute to explaining ϕ, then we can conclude anything—albeit only plainly.[32]

Finally, the discharge function D behaves as one would expect.[33]

Three observations about these rules. First, none of these principles ensure

[31] In view of this, it is significant that the putative counterexamples to transitivity in (Schaffer 2012) are all couched in terms of partial ground. For responses to the counterexamples see (Raven 2013; and Litland 2013).

[32] Note that (Non-circularity) does not simply take the form:

$$\begin{array}{c} \Gamma, \phi \\ \mathcal{E}(e) \\ \dfrac{\phi}{\psi}\ \text{Non-circularity*} \end{array}$$

For suppose that there is an explanatory argument from ϕ (and some further premisses to ϕ itself). Suppose further that ϕ follows plainly from Δ. We should be able to conclude anything from Δ, Γ. But since the argument from Δ to ϕ is merely plain if we use (Plain Chaining) we get the argument

$$\begin{array}{c} \begin{array}{cc} \Delta & \\ \phi & \quad \Gamma \end{array} \\ E(p) \\ \phi \end{array}$$

This argument is only plain, which means we cannot apply Non-circularity*. (Non-circularity) gets around this by building some chaining into the rule.

[33] To be pedantic. If ϕ is an instance of (Assumption) $D(\phi) = \phi$. In an instance of (Chaining) $D(\phi) = \bigcup_i D(\phi_i) \cup D(\Gamma)$, where $D(\Gamma)$ is the set of nodes decorated by the propositions in Γ. In an instance of (Non-circularity) $D(\psi) = \bigcup_i D(\delta_i) \cup D(\phi)$.

that there are any explanatory arguments: they only tell us how explanatory arguments combine to form further explanatory arguments. This is as it should be: it is not part of the job of a *pure* logic of ground to ensure that there are cases of strict ground.

Second, note that the principles in figure 1 respect the *non-monotonicity* of ground: if there is an explanatory argument from Δ to ϕ there need not be an explanatory argument from Δ, ψ to ϕ.

Third, the principles deal exclusively with explanatory arguments; no mention is made of explanatory *inference*. By developing a theory of explanatory inference we would be able to develop a logic of *immediate* ground. We refrain from doing so for two reasons. First, the previous discussions of iterated ground have focused on mediate ground; it is convenient to follow suit. Second, there are some technical problems in expressing that mediate ground is the closure of immediate ground under Cut. These problems can be solved, but the most convenient way of solving them involve both weak[34] and many-many ground.[35] Developing this machinery would be a distraction.

6.1 Is non-factive ground non-circular?

Explanatory arguments correspond to non-factive full ground in the sense that $\phi_0, \phi_1, \ldots$ non-factively ground ϕ iff there is an explanatory argument from $\phi_0, \phi_1, \ldots$ to ϕ. While (Non-circularity) ensures that factive ground is asymmetric, (Non-circularity) does not ensure that non-factive full ground is asymmetric. The reason is that the (Non-circularity) does not discharge the premisses on which ϕ depends.[36]

This is not an oversight: there is, arguably, nothing wrong with explanatory *arguments* from ϕ (and some further) premisses to ϕ itself. To see this consider the following situation. Suppose that a is part of b but it is possible that b instead is part of a. (a might be an organism that has entered into the body of organism b where a is now fulfilling some function inside b's body; but it might equally well have been b that had entered into the body of a.) Then, while the existence of a (partly) strictly grounds the existence of b, it is possible that the existence of b (partly) strictly grounds the existence of a. There would then be an explanatory argument from the claim that a exists (and some further premisses Γ_a) to the claim that b exists. But there is also an explanatory argument from the claim that b exists (and some further premisses Γ_b) to the claim that a exists. But then, by (Chaining), there is also an explanatory argument from the claim

[34] For the notion of weak ground, see (Fine 2012a, pp. 51–53, 2012b, pp. 3–4)

[35] Since the notion of many-many ground is of considerable interest it is of some importance that the zga can be extended to accomodate it. This can be done by treating arguments as certain directed, acyclic hypergraphs, where we now allow the head of an arc to be of any cardinality. This allows us to model inferences with several simultaneous conclusions—where the conclusions are read conjunctively. This differs from standard multiple-conclusion natural deduction—(see e.g., Read 2000)—where one reads the multiple conclusions disjunctively (as in multiple conclusion sequent calculus). I hope to return to these matters elsewhere.

[36] I am grateful to two anonymous reviewers for making me realize that I had to address the issues discussed in this and the next subsection.

that a exists (and the premisses Γ_a, Γ_b) to the claim that a exists.

While (Non-circularity) does not rule out arguments from ϕ (and some Γ) to ϕ itself, (Non-circularity) ensures that ϕ and Γ cannot jointly be the case. In particular, (Non-circularity) ensures that if there is an explanatory argument from just ϕ to ϕ itself then ϕ is not the case.

6.2 Impossible explanations

We have taken the line that in cases like the above we have a genuinely explanatory argument, it is just that it is impossible for its premisses to be jointly true. One might balk at this, insisting that for Γ to ground ϕ (even non-factively) it has to be possible for the propositions in Γ to be jointly true. If one does so insist, the above case shows that one has to reject (Chaining) for explanatory arguments. Should we so insist?

I think this would be a mistake. In fact, not only do I think that there are explanatory arguments with impossible premisses; I also think there are explanatory *inferences* with impossible premisses. Above I mentioned that I take conjunction introduction to result in explanatory inferences: this is so even in a case like $\dfrac{\phi \qquad \neg\phi}{\phi \wedge \psi}$ where both the premisses and the conclusion of the inference are impossible. The reason is that I take the explanatoriness of an inference to be a matter of the *form* of the inference; all inferences of that form are explanatory. It would take us too far afield fully to defend this view here, so let me offer a more concessive response. [37]

We could agree to reserve the words "explanatory argument" ("inference") for an argument (inference) where the premisses can be jointly true. (Arbitrarily chaining together explanatory inferences gives us the *shmexplanatory* arguments.) While there cannot be an explanatory argument from ϕ (and some premisses Γ) to ϕ itself there might be a shmexplanatory argument. What matters for my purposes is that if there is a shmexplanatory argument from ϕ (and Γ) to ϕ something has gone wrong and we can conclude that not all the propositions in $\Gamma \cup \{\phi\}$ are true. Non-factive ground in the present sense will

[37] Let me briefly mention some connections and issues. First, the notion of *form* is much broader than logical form narrowly construed. For instance, I hold that the inference $\dfrac{c \text{ is burgundy}}{c \text{ is red}}$ (where c is a particular name) is explanatory in virtue of its form. Secondly, there are important connections between the view that explanatoriness is a matter of form and the principle of *Formality* (Rosen 2010, pp. 131–132; see also Audi 2012b, pp. 697–698) and Fine's idea of "generic ground" (Fine 2014). Thirdly, for a particular inference, we obtain its *forms* (plural) by replacing particular constituents in the inference with schematic letters. The above burgundy–red inference, then has a form $\dfrac{x \text{ is burgundy}}{x \text{ is red}}$. Fourthly, a familiar point: an inference has many forms, and it need not be explanatory under all its form. (For instance, the burgundy–red inference is not explanatory under the form: $\dfrac{\Xi(x)}{\zeta(x)}$.) Can we make sense of a notion of a "canonical" form of an inference? And can the explanatoriness of a particular inference always be explained by reference to the explanatoriness of its canonical form? I hope to discuss these issues at greater length elsewhere. Thanks to Ralf Bader for discussion of these issues.

then be correlated not with explanatory arguments but with shmexplanatory arguments. Whenever a shmexplanatory argument has true premisses, however, it will be explanatory and so the relationship between factive ground and explanatory arguments is preserved.

I will continue to refer to explanatory arguments; if one has scruples one should mentally substitute "shmexplanatory argument" for "explanatory argument".

7 Grounding Operators: Introduction Rules

We can now give introduction rules for the non-factive grounding operator $\Rightarrow$. Since $\phi_0, \phi_1, \ldots \Rightarrow \phi$ is meant to report that there is an explanatory argument to ϕ from $\phi_0, \phi_1, \ldots$ its introduction-rule looks like this:

$$\dfrac{\dfrac{}{\phi_0, \phi_1, \ldots}\ 0,1,2,\ldots \\ \mathcal{E} \\ \phi}{\phi_0, \phi_1, \ldots \Rightarrow \phi}\ 0,1,2,\ldots, \Rightarrow\text{-Introduction}$$

Here $\mathcal{E}$ is an explanatory argument and $D(\phi)$ is labeled by all and only the propositions $\phi_0, \phi_1, \ldots$. In this case, we demand that we discharge all the propositions on which ϕ depends; that is, $D(\phi_0, \phi_1, \ldots \Rightarrow \phi) = \emptyset$. (Note the similarity to the introduction rule for a strict conditional: "$\Rightarrow$" stands to the explanatory arguments as the strict conditional stands to deductively valid arguments (cf. Scott 1971).)

We have to discharge *all and only* the premisses on which ϕ depends because $\Rightarrow$ is to capture non-factive *full* ground. If $\Delta \Rightarrow \phi$ is the case then Δ (if true) provides a full explanation for why ϕ is the case; moreover, every $\delta \in \Delta$ is relevant to explaining why ϕ is the case. This is captured by discharging all the $\delta \in \Delta$.

Since we are working with a deductive system with two types of argument it does not suffice to specify that arguments of a certain form are to be valid; we also have to specify whether arguments of the relevant form are to be explanatory or merely plain. We will treat arguments of the form depicted in the $\Rightarrow$-I rule as explanatory. What justifies us in so treating them?

The only reasonable alternative view would require more than an explanatory argument $\mathcal{E}$ from Δ to ϕ in order to allow us to conclude $\Delta \Rightarrow \phi$; in addition, one would require the premiss *that* $\mathcal{E}$ is explanatory. (If one adopted such a view the question would naturally arise what grounds the truth *that* the premiss $\mathcal{E}$ is explanatory.)

We should resist this view. What is needed to conclude $\Delta \Rightarrow \phi$ is just an explanatory argument $\mathcal{E}$ from Δ to ϕ; there is no need for the further truth *that* $\mathcal{E}$ is explanatory. The requirement that we need this further truth is as inappropriate as the demand that in order to apply conditional proof we need not just a valid argument $\mathcal{D}$ from ϕ (and some further premisses) to ψ, we need, in addition, the premiss *that* $\mathcal{D}$ is valid.

It might be helpful to think about this in terms of the machine picture. To determine whether $\Delta \Rightarrow \phi$ we go to a machine that encodes every explanatory inference. We then ask the machine to simulate the result of being fed input Δ. The machine then procedes to run the simulation. If the machine churns out ϕ it also churns out $\Delta \Rightarrow \phi$ and ends the simulation. At no step in this process is it necessary for the machine to check whether the inferences it carried out were explanatory.

Similarly, in order to check whether we can conclude the conditional $\phi \rightarrow \psi$ we might go to a machine that encodes all the logically valid inferences and ask it to simulate the result of being fed the proposition ϕ. If the machine churns out ψ, the machine stops its simulation and also churns out $\phi \rightarrow \psi$. At no point is it necessary for the machine to check whether the inferences it carried out are logically valid.

Having $\Rightarrow$ in place we can state introduction rules for the factive grounding operator <:

$$\frac{\Delta \qquad \Delta \Rightarrow \phi}{\Delta < \phi}\ \text{<-Introduction}$$

Arguments of this form, too, will be explanatory. Note the similarity to conjunction-introduction; note also that the following is an instance of <-introduction: $\dfrac{\Rightarrow \phi}{<\phi}$ <-I

In this setting there is nothing mysterious about the notion of zero-grounding. A truth ϕ is zero-grounded if there is an explanatory argument from the empty collection of premisses to the conclusion ϕ. If there is an explanatory argument $\mathcal{E}$ from Γ to ϕ, we can now show that the non-factive grounding claim $\Gamma \Rightarrow \phi$ is (factively and non-factively) zero-grounded.

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\overline{\;\Gamma\;}^{\,1}\\ \mathcal{E}\\ \phi\end{array}}{\Gamma \Rightarrow \phi}\ 1, \Rightarrow\text{-I}}{\Rightarrow (\Gamma \Rightarrow \phi)}\ \Rightarrow\text{-I}}{<(\Gamma \Rightarrow \phi)}\ \text{<-I}$$

We cannot yet show, however, that if $\Gamma \Rightarrow \phi$ is *true*, then $\Gamma \Rightarrow \phi$ is zero-grounded. To do this we need elimination rules for $\Rightarrow$.

8 Grounding Operators: Elimination Rules

We use a proof-theoretic inversion principle (see e.g., Read 2010) to find elimination rules. This principle says that the elimination rule(s) for an operator

λ should be such that if ϕ follows from each of the conditions (given by the introduction rules) allowing us to assert $\lambda(\psi_0, \ldots, \psi_n)$, then ϕ should follow from $\lambda(\psi_0, \ldots, \psi_n)$ by an elimination rule. And conversely: if ϕ follows from $\lambda(\psi_0, \ldots, \psi_n)$ by an elimination rule, then ϕ has to follow from any of the conditions allowing us to assert $\lambda(\psi_0, \ldots, \psi_n)$. (Where these latter conditions are given by the introduction rules.) Let us apply the principle to the easy case of $<$.

According to the $<$-introduction rule, we are entitled to conclude $\Delta < \phi$ from premisses Δ and $\Delta \Rightarrow \phi$. Anything that follows from those premisses must therefore follow from $\Delta < \phi$. The elimination rule for $<$ takes the form:

$$\dfrac{\Delta < \phi \qquad \begin{matrix} \overline{\Delta}^{\,1} & \overline{\Delta \Rightarrow \phi}^{\,2} \\ & \mathcal{E} \\ & \psi \end{matrix}}{\psi}\ 1{,}2\text{: <-Elimination}$$

This is to be read as follows. If $\mathcal{E}$ is an argument (explanatory or plain) to conclusion ψ and we, in the course of $\mathcal{E}$, have used the assumptions Δ and $\Delta \Rightarrow \phi$ some number of times, we can conclude ψ from $\Delta < \phi$, discharging any number of the assumptions Δ and $\Delta \Rightarrow \phi$.

Arguments of this form are plain. The premisses of an application of <-E do not explain its conclusion (or if they do this is not by dint of their being the premisses of an application of <-E). In general, the premisses of an application of an elimination rule do not explain its conclusion. We will therefore treat all elimination rules as giving rise to plain arguments.

The elimination rule for $\Rightarrow$ is more interesting; a second technical innovation is required to find elimination rules here. The introduction rule for $\Rightarrow$ tells us that we are entitled to assert $\Delta \Rightarrow \phi$ if there is an explanatory argument with premisses (all and only) Δ and conclusion ϕ. Anything that follows from the existence of such an argument should follow from $\Delta \Rightarrow \phi$. How can we express the assumption that there *exists* an explanatory argument from premisses Δ to conclusion ϕ?

We cannot—not without extending our conception of natural deduction: we have to be able to assume and discharge *arguments* as well as propositions. To do this we introduce the notion of a *hypothetical argument*.

An expression of the form $\Delta \Vdash_e \phi$ is to stand for a hypothetical explanatory argument with conclusion ϕ and premisses (exactly) Δ.[38] Such hypothetical arguments $\Delta \Vdash_e \phi$ only occur in contexts of the form:

$$\frac{\Delta}{\phi}\,[\Delta \Vdash_e \phi]$$

[38]For an application of, in effect, hypothetical arguments in a different context see Schroeder-Heister 1984.

Arguments of this form are explanatory. What this says is that by using a hypothetical explanatory argument from Δ to ϕ one explanatorily infers ϕ from Δ.

Even though hypothetical arguments can be assumed and discharged in the course of an argument, they are not premisses on which the (sub)conclusions depend. (We assume an argument, not the existence of an argument.) One may think of the hypothetical arguments as assumed *rules* of inference. If ϕ follows from ψ by a rule of inference R, then if R were not valid, ϕ would not have followed from ψ, but that does not mean that the *validity* of the rule R is a premiss which has to be added to ψ in order to derive ϕ. (We assume the rule, not the validity of the rule.)[39]

We can now write down the elimination rule for $\Rightarrow$:

$$\dfrac{\Delta \Rightarrow \phi \qquad \begin{array}{c} \overline{\Delta \Vdash_e \phi}^{\,1} \\ \mathcal{E} \\ \psi \end{array}}{\psi}\ 1, \Rightarrow\text{-Elimination}$$

This is read as follows. Suppose we have an argument $\mathcal{E}$ with conclusion ψ in the course of which we have relied on some instances of the hypothetical argument $\Delta \Vdash_e \phi$. $\Rightarrow$-E allows us to conclude ψ from $\Delta \Rightarrow \phi$ discharging any number of occurrences of the hypothetical argument $\Delta \Vdash_e \phi$.

For any given collection of explanatory and plain arguments $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ we define the explanatory and plain arguments over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ as the least class of arguments $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ such that $\mathfrak{E}_e \subseteq \mathfrak{E}'_e$ and $\mathfrak{E}_p \subseteq \mathfrak{E}'_p$ and such that $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ is closed under the rules in figure 1 and the introduction and elimination rules given above. The arguments of the Pure Logic of Iterated Strict Full Ground (PLISFG) are the explanatory and plain arguments over $\langle \emptyset, \emptyset \rangle$.

We say that ϕ follows from Γ in the Pure Logic of Iterated Strict Full Ground (PLISFG) if there is $\Gamma_0 \subseteq \Gamma$ such that there is an argument in PLISFG with premisses Γ_0 and conclusion ϕ. We write $\Gamma \vdash \phi$ if this is the case. PLISFG suffices to prove (almost) all the basic facts about non-iterated factive ground.[40]

Proposition 8.1. *(i) Left factivity:* $\Delta < \phi \vdash \delta$, *for all* $\delta \in \Delta$.

(ii) Right factivity: $\Delta < \phi \vdash \phi$.

(iii) Non-circularity: $(\Delta, \phi) < \phi \vdash \psi$ *for all* ψ.

[39]Two reasons for insisting that we assume the argument and not its validity: first, if we had to assume the validity of the argument $\Delta \Vdash_e \phi$ in addition to the argument $\Delta \Vdash_e \phi$ we would be off on a regress à la (Carroll 1895). (In fact this regress was noted already in 1837! (Bolzano 1972, vol. 2, § 199).) Second, the idea behind the hypothetical arguments is that to assume $\Delta \Vdash_e \phi$ is to assume that we can explain ϕ from Δ; if we had to assume the validity of the argument $\Delta \Vdash_e \phi$ and not just the argument $\Delta \Vdash_e \phi$ this would not work. For suppose we derive ϕ from Δ using the hypothetical argument $\Delta \Vdash_e \phi$, and that ϕ now depends not just on Δ but also on the validity of $\Delta \Vdash_e \phi$. We would now not have an explanation of ϕ from Δ; we would rather have an explanation of ϕ from Δ together with the validity of $\Delta \Vdash_e \phi$.

[40]It does not prove the principle of Amalgamation. See appendix A for a way of fixing this.

(iv) *Cut:* $\Delta_0 < \phi_0, \Delta_1 < \phi_1, \ldots, (\phi_0, \phi_1, \ldots, \Gamma < \phi) \vdash \Delta_0, \Delta_1, \ldots, \Gamma < \phi$.

Proof: Given the unfamiliarity of the system let us prove Proposition 8.1 (ii) and Proposition 8.1 (iii), leaving the rest as an exercise for the reader. (To prove Proposition 8.1 (i) and Proposition 8.1 (iv) we require vacuous discharge.)

$$\dfrac{\Delta < \phi \qquad \dfrac{\dfrac{}{\Delta \Rightarrow \phi}\,{\scriptstyle 1} \qquad \dfrac{\dfrac{}{\Delta}\,{\scriptstyle 2}}{\phi}\,[\Delta \Vdash_e \phi]^3}{\phi}\,{\scriptstyle 3,\ \Rightarrow\text{-E}}}{\phi}\,{\scriptstyle 1,2,\ <\text{-E}}$$

$$\dfrac{\Delta, \phi < \phi \qquad \dfrac{\dfrac{}{\Delta, \phi \Rightarrow \phi}\,{\scriptstyle 1} \qquad \dfrac{\dfrac{\dfrac{}{\Delta, \phi}\,{\scriptstyle 2}}{\phi}\,[\Delta, \phi \Vdash_e \phi]^3}{\psi}\,\text{Non-circularity}}{\psi}\,{\scriptstyle 3,\ \Rightarrow\text{-E}}}{\psi}\,{\scriptstyle 1,2\ <\text{-E}}$$

We can, moreover, prove the following two crucial principles about iterated ground. (For readability, we write $\emptyset \Rightarrow \phi$ instead of $\Rightarrow \phi$; and similarly for $<$.)

Proposition 8.2. *(i)* $\Delta \Rightarrow \phi \vdash \emptyset < (\Delta \Rightarrow \phi)$

(ii) $\Delta < \phi \vdash \Delta < (\Delta < \phi)$

Proof: The following derivation establishes Proposition 8.2 (i):

$$\dfrac{\Delta \Rightarrow \phi \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta}\,{\scriptstyle 1}}{\phi}\,[\Delta \Vdash_e \phi]^2}{\Delta \Rightarrow \phi}\,{\scriptstyle 1,\ \Rightarrow\text{-I}}}{\emptyset \Rightarrow (\Delta \Rightarrow \phi)}\,{\scriptstyle \Rightarrow\text{-I}}}{\emptyset < (\Delta \Rightarrow \phi)}\,{\scriptstyle <\text{-I}}}{\emptyset < (\Delta \Rightarrow \phi)}\,{\scriptstyle 2,\ \Rightarrow\text{-E}}$$

Note here how the hypothetical argument $\Delta \Vdash_e \phi$ (labeled "2") is not counted as a premiss on which the sub-conclusion ϕ depends. This is required in order for the applications of $\Rightarrow$-I to be justified: we have to discharge *all* assumptions on which ϕ depends. Note also how we use $<$-I in the case where Δ is empty.

The following establishes Proposition 8.2 (ii):

$$\dfrac{\Delta < \phi \qquad \dfrac{\overline{\Delta \Rightarrow \phi}^{\,2} \qquad \dfrac{\overline{\Delta}^{\,1} \qquad \dfrac{\dfrac{\overline{\Delta}^{\,5} \qquad \dfrac{\dfrac{\overline{\Delta}^{\,3}}{\phi}\,[\Delta \Vdash_e \phi]^4}{\Delta \Rightarrow \phi}\,{\scriptstyle 3,\,\Rightarrow\text{-I}}}{\Delta < \phi}\,{\scriptstyle <\text{-I}}}{\Delta \Rightarrow (\Delta < \phi)}\,{\scriptstyle 5,\,\Rightarrow\text{-I}}}{\Delta < (\Delta < \phi)}\,{\scriptstyle <\text{-I}}}{\Delta < (\Delta < \phi)}\,{\scriptstyle 4,\Rightarrow\text{-E}}}{\Delta < (\Delta < \phi)}\,{\scriptstyle 1,2\ <\text{-E}}$$

These results show that factive ground (<) behaves as it should; in particular, all the constraints on factive ground laid down in § 2 above are satisfied by <.[41]

For more technical detail about PLISFG see appendices A and B.

9 Every (true) non-factive grounding claim has the same ground

The following objection comes naturally to mind. "According to the ZGA every (true) non-factive grounding claim has the same ground—indeed, the same *immediate* ground—namely the empty one. How could this be? Clearly, the explanation for why $p \Rightarrow p \wedge p$ is the case differs from the explanation for why $p \Rightarrow p \vee p$ is the case?"[42]

This criticism is misguided: while every true non-factive grounding claim $\Delta \Rightarrow \phi$ has the same (immediate) strict full ground—the empty ground—different true non-factive grounding claims are (immediately) zero-grounded in different *ways*. This is easiest to see by looking at the machine picture. When the machine is asked to simulate being fed p it will churn out both $p \Rightarrow p \wedge p$ and $p \Rightarrow p \vee p$. But there is no reason to think that the simulations the machine runs in order to reach these outputs are the same; in particular, the two simulations might comprise the application of different mechanisms.

In the framework of the explanatory arguments we can see that the objection trades on an ambiguity in what is meant by "explanation". In this framework there are two things one can mean by an explanation of ϕ: one can mean a collection of propositions $\phi_0, \phi_1, \ldots$ from which ϕ can be derived in an explanatory way; alternatively, one can mean an *argument* witnessing that ϕ can be derived in an explanatory way. In the former sense $p \Rightarrow p \wedge p$ and $p \Rightarrow p \vee p$ have the same explanation; in the latter sense they have different explanations.

To see this consider figure 2. This depicts the arguments establishing that $\Rightarrow (p \Rightarrow p \wedge)$ and $\Rightarrow (p \Rightarrow p \vee p)$. These arguments differ, amongst other things, in that the first contains an application of $\wedge$-introduction while the second contains

[41] Note that we do *not* have $(\Gamma, \phi \Rightarrow \phi) \vdash \psi$, for each ψ. This, as we argued in § 6.1 above, is as it should be.

[42] (Dasgupta 2014c, p. 573) criticizes the views of (deRosset 2013) and (Bennett 2011) along these lines.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p}\,{}^{1} \qquad \dfrac{}{p}\,{}^{1}}{p \wedge p}\,\wedge\text{-I}}{p \Rightarrow p \wedge p}\,1, \Rightarrow\text{-I}}{\Rightarrow (p \Rightarrow p \wedge p)}\,\Rightarrow\text{-I} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{p}}\,{}^{1}}{p \vee p}\,\vee\text{-I}}{p \Rightarrow p \vee p}\,1,\Rightarrow\text{-I}}{\Rightarrow (p \Rightarrow p \vee p)}\,\Rightarrow\text{-I}$$

Figure 2: Different ways

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p}\,{}^{1} \qquad \dfrac{}{q}\,{}^{2}}{p \wedge q}\,\wedge\text{-I}}{p, q \Rightarrow p \wedge q}\,1,2, \Rightarrow\text{-I}}{\Rightarrow (p, q \Rightarrow p \wedge q)}\,\Rightarrow\text{-I} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{r}\,{}^{1} \qquad \dfrac{}{s}\,{}^{2}}{r \wedge s}\,\wedge\text{-I}}{r, s \Rightarrow r \wedge s}\,1,2, \Rightarrow\text{-I}}{\Rightarrow (r, s \Rightarrow r \wedge s)}\,\Rightarrow\text{-I}$$

Figure 3: Same way

instead an application of $\vee$-introduction. (We assume that $\vee$-introduction and $\wedge$-introduction are rules of explanatory inference.)

What is depicted in figure 2 is a situation where two distinct propositions have exactly the same grounds but are grounded in different ways. But there are also situations where distinct propositions have the same grounds *and* are grounded in the same way. Figure 3 depicts such a situation. In this case not only do $\Rightarrow (p, q \Rightarrow p \wedge q)$ and $\Rightarrow (r, s \Rightarrow r \wedge s)$ have the same ground—the empty one—but they also seem to be grounded in the empty ground in *the same way*. We can make sense of this in terms of the machine picture. In the two cases depicted in figure 3 the same *mechanisms* are applied (and they are applied in the same order) but the *applications* of the mechanisms differ since p, q are different propositions from r, s. (It is for this reason that we have taken an arc A to represent the *application* of a mechanism not the mechanism itself.)

An advantage of the framework of explanatory arguments is that it promises us the means for defining the notion of a way of grounding. For consider a particular explanatory argument. Uniformly replace items in that argument with schematic letters. This gives us an argument *form*. If every argument of that form is explanatory the argument form is an *explanatory* argument form. We may identify the ways of grounding with the explanatory argument forms. As is easily seen, the two arguments in figure 3 have the same explanatory form.

We can now deal with a further objection to the ZGA. Dasgupta (2014c, pp. 531–2) observes that there are patterns in grounding. (For instance, all conjunctions are alike in terms of how they are grounded.) An account of ground should provide us with an account of these patterns. Armed with the notion of a way of grounding we have a very simple explanation: the patterns in grounding are the result of different facts being grounded in the same way.[43]

[43] I believe the notion of a *way* of ground is of great importance both for the theory of ground and for the applications of ground. More has to be done to put the talk of ways of grounding on a

10 Comparison with the Straightforward Account

The SFA, recall, holds that when $\Gamma < \phi$, then $\Gamma < (\Gamma < \phi)$; unlike the ZGA, however, the SFA does not hold that $\Gamma < \phi$ is partly grounded in $\Gamma \Rightarrow \phi$. Why should we prefer the ZGA to the SFA? The question has a false presupposition: on my favored way of conceiving of the relationship between the SFA and the ZGA they are not in competition.

Let us first observe that the logical techniques developed for the ZGA can equally well be employed in developing a logic of iterated ground that is in accord with the SFA. A defender of the SFA could, e.g., give an introduction rule for an operator $<^+$ as follows:

$$\dfrac{\Delta \qquad \begin{array}{c}\overline{\Delta}^{\,1}\\ \mathcal{E}\\ \phi\end{array}}{\Delta <^+ \phi}\ 1, <^+\text{-I}$$

Here $\mathcal{E}$ is an explanatory argument and Δ are all and only the premisses on which ϕ depends. This rule in effect compresses the $\Rightarrow -I$ and <-I rules into one rule.[44] Is there something wrong such an introduction rule for $<^+$?

In my view there is nothing wrong with an operator governed by such an introduction rule. We should just insist that the operators $<^+$ and $<$ capture different notions of factive ground. The propositions $\Delta < \phi$ and $\Delta <^+ \phi$ are, after all, readily distinguished in terms of their grounds: $\Delta < \phi$, but not $\Delta <^+ \phi$, is partially grounded in $\Delta \Rightarrow \phi$. We may also describe the difference in terms of explanatory arguments: while one *gives* an explanatory argument from Δ to ϕ as part of explanatorily inferring $\Delta <^+ \phi$, the claim *that* there is such an explanatory argument (that is: $\Delta \Rightarrow \phi$) is not part of the grounds for $\Delta <^+ \phi$.

One might, however, worry that the possibility of introduction rules like the ones for $<^+$ above present a problem for the ZGA: could one, by holding that $<^+$ was the "real" notion of factive ground, give a satisfactory solution to the Status Problem without invoking either non-factive ground or zero-grounding?

This would be the wrong conclusion to draw—for two reasons. First, if one accepts the framework of explanatory arguments it is hard to reject the notion

rigorous footing, but this is not the place to do this. The issues that arise are very closely related to the issues of *form* mentioned in footnote 37.

[44] One might wonder why the rule should not be given as follows instead:

$$\dfrac{\begin{array}{c}\Delta\\ \mathcal{E}\\ \phi\end{array}}{\Delta <^+ \phi}\ 1, <^+\text{-I}$$

(Here Δ are all and only the premisses on which ϕ depends and $\mathcal{E}$ is explanatory.) The difference between the two rules is that in the latter one, Δ is not discharged. The reasons for this are purely technical: it makes finding elimination rules easier and it avoids certain complications in the statement of (Chaining).

of non-factive ground. And if one accepts the notion of non-factive ground, a Status Problem like the one in § 3 arises for non-factive ground. This presents the defender of the SFA with a challenge: if true non-factive grounding claims are not zero-grounded, in what are they grounded?

Second, it turns out that even if we reject the notion of non-factive ground and treat factive ground as $<^{+}$ we still need zero-grounding. To see this, consider *negated* grounding claims.[45]

Suppose that the object a occurs in the truth ϕ and that while each $\delta \in \Delta$ is the case, $\neg(\Delta < \phi)$ is also the case. (For a concrete example: let a be Socrates, ϕ the truth that Socrates was Greek, and Δ the truth that the Holy Roman Empire was dissolved in 1806.) If $\neg(\Delta < \phi)$ is true and ungrounded then the object a would be O-fundamental. If the Status Problem is to be satisfactorily solved, true negated grounding claims, too, have to be grounded.

The defenders of SFA have had nothing to say about such negated grounding claims.[46] Let us first consider what a defender of the ZGA could say. For a defender of the ZGA the immediate full grounds of $\Delta < \phi$ are Δ and $\Delta \Rightarrow \phi$ taken together. It is plausible to hold that to ground $\neg(\Delta < \phi)$ is either to ground $\neg\delta$ for a $\delta \in \Delta$ or to ground $\neg(\Delta \Rightarrow \phi)$.[47] In the envisaged scenario, each $\delta \in \Delta$ is true; what has to be grounded, then, is $\neg(\Delta \Rightarrow \phi)$.

The natural move is to say that such true negated non-factive grounding claims are zero-grounded. One is, to be sure, owed a justification for treating such negated non-factive grounding claims as zero-grounded; but the machine and graph-theoretical pictures of § 5 provide a natural justification.

In terms of the machine-picture: when the machine is fed no input, it simulates being given the propositions Δ as input. The machine then tries to find an arc from Δ to ϕ. When it has tested *all* the arcs A with $t(A) = \Delta$ and has noted that none of them have $h(A) = \{\phi\}$, the machine reports back that $\neg(\Delta \Rightarrow \phi)$. Since the machine was fed no input $\neg(\Delta \Rightarrow \phi)$ will be zero-grounded.[48]

If one accepts the connection between grounding and explanatory arguments developed in this paper one would have to exhibit natural introduction and elimination rules for *negated* non-factive grounding claims ensuring that true non-factive grounding claims are zero-grounded. This can in fact be done but we cannot go into the details here.[49]

[45] I should stress that negated—more generally, embedded—grounding claims cause problems even if one does not accept the present framework of explanatory arguments.

[46] As admitted by (deRosset 2013, p. 16).

[47] The simplest way of dealing with negation in the logic of ground is to follow a broadly "bilateralist" strategy and give separate introduction and elimination rules for negated and unnegated propositions; for this strategy, see e.g., (Fine 2012a, p. 63). A different, perhaps preferable, treatment of negation is pursued in (Fine 2014).

[48] How does the machine know *that* it has inspected all the arcs? We might imagine that in running the simulation the arcs are ordered in such a way that each arc occurs infinitely often but such that if an arc A with tail Δ occurs both at position t and at a later position t', then all arcs with tail Δ have occurred at least once before t'. If, having found no arc B with tail Δ and head ϕ, the machine inspects an arc it has previously inspected, it aborts the simulation and outputs $\neg(\Delta \Rightarrow \phi)$.

[49] In this paper we have limited ourselves to simple hypothetical *arguments* from some propositions to a proposition. To give a rule for negated grounding claims we would have to be able

One can say something similar about $<^+$: if Δ is true and $\neg(\Delta <^+ \phi)$ is also the case, then $\neg(\Delta <^+ \phi)$ is (immediately) zero-grounded. In terms of the machine picture the account is much the same. The machine is fed no input and simulates the result of being fed Δ as input. It runs through all arcs with tail Δ. After observing that there is no arc with tail Δ and head ϕ, the machine churns out the truth $\neg(\Delta <^+ \phi)$.

Holding that factive ground behaves like $<^+$ does not obviate the need for zero-grounding. [50]

11 Concluding Remarks

In this paper I have developed a novel account of iterated grounding—the ZGA—and I have showed how—by taking a notion of explanatory argument as basic one can develop a logic of iterated ground. No matter what one thinks of the metaphysics of the ZGA, the logical techniques developed here should prove useful for others interested in the logic of ground. In closing let me mention some areas where more work needs to be done.

First, more needs to be said about explanatory *inference* (as opposed to argument). Explanatory inference is of interest because of its connection to *immediate* ground and having a logic of immediate ground would be of considerable interest.

Second, more needs to be said about the notions of a *way* of grounding and the *form* of an explanatory argument. I believe that it is only by geting clearer on these notions that we, in a principled manner, can develop fine-grained conceptions of ground—ones, e.g., where a proposition p strictly grounds the conjunction $p \wedge p$.

Third, we have restricted our attention to the *pure* logic of ground. To fully vindicate the framework we have to be able to extend the framework of explanatory arguments to deal with the impure logic of ground. Of particular philosophical interest is extensions dealing with negation and the conditional. The case for the ZGA would be greatly strengthened if this could be done in a natural way.

Fourth, more must be said about the relationship between essence and ground. Many philosophers have held that there is a deep connection between essence and ground.[51] I believe that the ZGA allows us to adopt a distinctive view on relationship between essence and ground; I end by baldly stating that

to assume *rules*. In particular, if we know that $\neg(\Delta \Rightarrow \phi)$ we should be entitled to assume a rule that lets us conclude the absurd from any explanatory argument from Δ to ϕ. That is, we have to assume not just arguments that allow us to pass from (sets of) propositions to propositions, but also rules that allow us to pass from arguments (or more generally: rules) to propositions. By extending the machinery of higher-order rules in (Schroeder-Heister 1984) to a setting where we have both explanatory and plain arguments this can be done.

[50] Thanks to an anonymous reviewer for pressing me to get clearer on the relationship between the ZGA and the SFA.

[51] Their number include: (Audi 2012b; Fine 2012a; Trogdon 2013b). (Rosen 2010; and Dasgupta 2014c) have even held that truths about ground are always partly *grounded* in truths about essence.

view.

Those who accept that there is a connection between essence and ground have held that it is certain *truths* about grounding that are essentially true. For instance, Rosen (2010, p. 130) holds that it is essential to disjunction that for all propositions p, if p is true then p grounds $p \vee q$. In my view this is a mistake: what is essential to disjunction is not a *truth*, but rather an explanatory *inference*. (In this case, the explanatory inferences from p to $p \vee q$ and from p to $q \vee p$.)[52]

References

Audi, Paul (2012a). "A Clarification and Defense of the Notion of Grounding". In: *Metaphysical Grounding*. Ed. by Fabrice Correia and Benjamin Schnieder. Cambridge University Press, pp. 101–121.

— (2012b). "Grounding: towards a theory of the in virtue of relation". In: *Journal of Philosophy* 109.12, pp. 685–711.

Bennett, K. (2011). "By Our Bootstraps". In: *Philosophical Perspectives* 25.1, pp. 27–41.

Bolzano, Bernhard (1972). *Theory of Science. Attempt at a Detailed and in the main Novel Exposition of Logic with Constant Attention to Earlier Authors*. Trans. by Rolf George. University of California Press.

Carroll, L. (1895). "What the tortoise said to Achilles". In: *Mind* 4.14, pp. 278–280.

Correia, Fabrice (2010). "Grounding and Truth-functions". In: *Logique et Analyse* 53.211, pp. 251–279.

— (2012). "On the Reduction of Necessity to Essence". In: *Philosophy and Phenomenological Research* 84.3, pp. 639–653.

— (2014). "Logical Grounds". In: *Review of Symbolic Logic* 7.1, pp. 31–59.

Dasgupta, Shamik (2014a). "Metaphysical Rationalism". In: *Nous*.

— (2014b). "On the Plurality of Grounds". In: *Philosopher's Imprint* 14.20, pp. 1–28.

— (2014c). "The Possibility of Physicalism". In: *Journal of Philosophy* 111.9/10, pp. 557–592.

deRosset, Louis (2013). "Grounding Explanations". In: *Philosophers' Imprint* 13.7, pp. 1–26.

— (forthcoming). "Better Semantics for the Pure Logic of Ground". In: *Analytic Philosophy*.

Dixon, T. Scott (forthcoming). "What is the Well-Foundedness of Grounding?" In: *Mind*.

[52] The idea that what is essential might be an inference is not unprecedented: Fine suggested that we should think of "the nature of the logical concepts as being given, not by certain logical truths, but by certain inferences." (Fine 1994, pp. 57-58) The idea is developed in greater detail in (Correia 2012). What is new in the present setting is the distinction between explanatory and merely plain inference and the claim that what is essential are certain *explanatory* inferences.

Fine, Kit (1994). "Senses of Essence". In: *Modality, Morality , and Belief: Essays in Honor of Ruth Barcan Marcus*. Ed. by N. Ascher D. Raffman W. Sinnott-Armstrong. Chicago: University of Chicago Press, pp. 53–73.

— (2001). "The Question of Realism". In: *Philosophers Imprint* 1.1, pp. 1–30.

— (2010a). "Some Puzzles of Ground". In: *Notre Dame J. Formal Logic* 51.1, pp. 97–118.

— (2010b). "The Question of Ontology". In: *Metametaphysics: new essays on the foundations of ontology*. Ed. by D.J. Chalmers, D. Manley, and R. Wasserman. Clarendon Press. Chap. 5, pp. 157–177.

— (2012a). "Guide to Ground". In: *Metaphysical Grounding*. Ed. by Fabrice Correia and Benjamin Schnieder. Cambridge University Press. Chap. 1, pp. 37–80.

— (2012b). "The Pure Logic of Ground". In: *The Review of Symbolic Logic* 5.1, pp. 1–25.

Fine, Kit (2014). "Identity criteria and ground". In: *Philosophical Studies*, pp. 1–19.

Fine, Kit (2014). "Truth-maker Semantics for Intuitionistic Logic". In: *Journal of Philosophical Logic* 43.2-3, pp. 549–577.

Jenkins, C. (2011). "Is Metaphysical Dependence Irreflexive?" In: *The Monist* 94.2, pp. 267–276.

Krämer, Stephan (2013). "A Simpler Puzzle of Ground". In: *Thought* 2.2, pp. 85–89.

Litland, Jon Erling (2013). "On Some Counterexamples to the Transitivity of Grounding". In: *Essays in Philosophy* 14.1: *Grounding Relation(s)*. Ed. by Paul Hovda and Troy Cross, pp. 19–32.

— (forthcoming[a]). "An infinitely descending chain of ground without a lower bound". In: *Philosophical Studies*, pp. 1–9.

— (forthcoming[b]). "Pure Logic of Many-Many Ground". In: *Journal of Philosophical Logic*.

Poggiolesi, Francesca (forthcoming). "On Defining the Notion of Complete and Immediate Formal Grounding". In: *Synthese*, pp. 1–21.

Rabin, Gabriel and Brian Rabern (2015). "Well-founding Grounding Grounding". In: *Journal of Philosophical Logic*.

Raven, Michael J. (2009). "Ontology, From a Fundamentalist Point of View". PhD thesis. New York University.

— (2013). "Is Ground a Strict Partial Order". In: *American Philosophical Quarterly* 50.2, pp. 191–199.

Read, Stephen (2000). "Harmony and Autonomy in Classical Logic". In: *Journal of Philosophical Logic* 29, pp. 123–154.

— (2010). "General-elimination harmony and the meaning of the logical constants". In: *Journal of Philosophical Logic* 39.5, pp. 557–576.

Rosen, G. (2010). "Metaphysical Dependence: Grounding and Reduction". In: *Modality: Metaphysics, Logic, and Epistemology*. Oxford University Press, pp. 109–135.

Schaffer, Jonathan (2009). "On what grounds what". In: *Metametaphysics: new essays on the foundations of ontology*. Ed. by D.J. Chalmers, D. Manley, and R. Wasserman. Oxford University Press, pp. 347–83.
— (2012). "Grounding, Transitivity, and Contrastivity". In: *Metaphysical Grounding*. Ed. by Fabrice Correia and Benjamin Schnieder. Cambridge University Press. Chap. 4, pp. 122–138.
— (forthcoming). "Grounding in the Image of Causation". In: *Philosophical Studies* 1-52.
Schnieder, Benjamin (2011). "A Logic for "Because"". In: *Review of Symbolic Logic* 4.3, pp. 445–465.
Schroeder-Heister, Peter (1984). "A natural extension of natural deduction". In: *Journal of Symbolic Logic* 49, pp. 1284–1300.
Scott, Dana (1971). "On Engendering an Illusion of Understanding". In: *The Journal of Philosophy*, pp. 787–807.
Sider, T. (2012). *Writing the book of the World*. Oxford University Press.
Trogdon, Kelly (2013a). "An Introduction to Grounding". In: *Varieties of Dependence*. Ed. by M. Hoeltje, A. Steinberg, and B. Schnieder. Philosophia Verlag, pp. 97–122.
— (2013b). "Grounding: Necessary or Contingent?" In: *Pacific Philosophical Quarterly* 94.4, pp. 465–85.
Wilson, Jessica M. (2014). "No Work for a Theory of Grounding". In: *Inquiry* 57.5-6, pp. 535–579.

$$\frac{\Delta, \phi < \phi}{\perp}\ \text{Non-circularity}$$

$$\frac{\Delta_0 < \phi_0 \qquad \Delta_1 < \phi_1 \ldots \qquad \phi_0, \phi_1, \ldots, \Gamma < \phi}{\Delta_0, \Delta_1, \ldots, \Gamma < \phi}\ \text{Cut}$$

$$\frac{\Delta_0 < \phi \qquad \Delta_1 < \phi \ldots}{\Delta_0, \Delta_1, \ldots < \phi}\ \text{Amalgamation}$$

Figure 4: Pure Logic of Strict Full Ground

A plisfg and plsfg

The Pure Logic of Strict Full Ground—plsfg—is the subsystem of Fine's Pure Logic of Ground—plg—that concerns only strict full ground. Its rules are depicted in figure 4.

plsfg is a *sequent* system: in plsfg an expression of the form $\Gamma < \phi$ is a sequent not a sentence and so a claim of iterated ground would be ill-formed.

There is an obvious translation of plsfg into the language of plisfg: we translate the sequent $\Delta < \phi$ by the formula $\Delta < \phi$. Proposition 8.1 ensures that plisfg proves the translations of Non-circularity and Cut. It turns out that we cannot derive the translation of the Amalgamation principle. In fact, Amalgamation is arguably not correct. To see this, consider the following situation. Suppose the inferences from ψ (and θ) to $\psi \vee \theta$ are explanatory; and suppose that these are the only explanatory inferences with conclusion $\psi \vee \theta$. Suppose further that there is an explanatory inference from ϕ to ψ but no explanatory inference from ϕ to θ. Then there is an explanatory argument from ϕ to $\psi \vee \theta$ and an explanatory argument from ψ to $\psi \vee \theta$, but there is no explanatory argument from ϕ, ψ to $\psi \vee \theta$. So while $\phi \Rightarrow \psi \vee \theta$ and $\psi \Rightarrow \psi \vee \theta$ we do not have $\phi, \psi \Rightarrow \psi \vee \theta$.[53,54]

To facilitate comparison with Fine's system we will, however, insist on Amalgamation. We validate Amalgamation by using "amalgamation-friendly" $\Rightarrow$ rules—see figure 5. Say that $\Delta_0, \Delta_1, \ldots, i \in I$ is a covering of Δ if $\bigcup_{i \in I} \Delta_i = \Delta$. In the amalgamation-friendly $\Rightarrow$-E rule the $i_0, i_1, \ldots, j_0, j_1, \ldots$ enumerate the *coverings* of Δ.

We now (re)define plisfg as follows. The *arguments* of plisfg are the least classes of explanatory and plain arguments closed under the constraints in

[53] Correia (2014, n. 17) makes the same observation, using a different example.

[54] Not all applications of Amalgamation are problematic. If ϕ, ψ are as above, we should have $\phi, \psi \Rightarrow \psi \wedge \psi$. The simplest way of getting a satisfactory treatment is by having the grounding operators take multisets on the left. Then since $\psi, \psi \Rightarrow \psi \wedge \psi$ we can derive $\phi, \psi \Rightarrow \psi \wedge \psi$ using Cut; in contrast, since we do not have $\psi, \psi \Rightarrow \psi \vee \theta$ we cannot derive $\phi, \psi \Rightarrow \psi \vee \theta$ using Cut.

$$
\dfrac{\begin{array}{cc} \Delta_0 & \Delta_1 \ldots \\ \mathcal{E}_0 & \mathcal{E}_1 \ldots \\ \phi & \phi \ldots \end{array}}{\Delta_0, \Delta_1, \ldots \Rightarrow \phi} \Rightarrow\text{-I}
$$

$$
\dfrac{\Delta \Rightarrow \phi \qquad \begin{array}{cc} \dfrac{}{\Delta_{i_0} \Vdash_e \phi}\, i_0 & \dfrac{}{\Delta_{i_1} \Vdash_e \phi \quad \ldots}\, i_1 \ldots \\ & \psi \ldots \end{array} \qquad \begin{array}{cc} \dfrac{}{\Delta_{j_0} \Vdash_e \phi}\, j_0 & \dfrac{}{\Delta_{j_1} \Vdash_e \phi \ldots}\, j_1 \ldots \\ \psi \ldots & \end{array}}{\psi} \; i_0, i_1, \ldots, j_0, j_1, \ldots, \Rightarrow\text{-E}
$$

Figure 5: Amalgamation-Friendly Rules.

figure 1, the amalgamation-friendly $\Rightarrow$ rules and the previous rules for $<$. Let $\vdash$ be the provability relation in PLISFG and let $\vdash_{\text{PLSFG}}$ be the provability relation in PLSFG.

It is easy to show that the translation of the rules of PLSFG are derivable in PLISFG: PLISFG is an extension of PLSFG. We can show that PLISFG is a conservative extension of the (relevant subsystem of) Fine's Pure Logic of Ground. This provides some support for the ZGA. Since it is plausible that PLSFG is the correct logic of strict full ground, this conservativeness result shows that by accepting the ZGA one is not thereby forced to accept any claims of non-iterated ground that one was not independently committed to accepting.

In order to establish this conservativeness result we develop a graph-theoretic semantics for PLSFG and show that any model for PLSFG not verifying a certain sequent $\Delta < \phi$ can be extended to a model for PLISFG not verifying the corresponding statement $\Delta < \phi$.[55]

B Graphical Semantics

The semantics will be based on the directed, pointed hypergraphs introduced in § 5. We impose the following demands on the hypergraphs.

A directed hypergraph $\mathcal{G} = \langle V, \mathcal{A}, t, h \rangle$ is *chained* if for all arcs $B, A_0, A_1, \ldots$ such that $t(B) = \{v_0, v_1, \ldots, w_0, w_1, \ldots\}$ and $h(A_i) = \{v_i\}$, for each i, there is an arc

[55] I should stress that the graph-theoretic semantics has some artificial features—for instance, the treatment of $<$ and $\Rightarrow$ is "written in by hand". I hope to provide a more illuminating semantics elsewhere; for the purposes of establishing the conservativity results these artificial features are, in any case, unproblematic.

C with $t(C) = t(A_0) \cup t(A_1) \cup \cdots \cup \{w_0, w_1, \dots\}$ and $h(B)$.[56]

A directed hypergraph $\mathcal{G} = \langle V, \mathcal{A}, t, h \rangle$ is *amalgamating* if whenever $\{A_i\}_{i \in I} \subset \mathcal{A}$ are such that $h(A_i) = h(A_j)$, for all $i, j \in I$ there is an $A \in \mathcal{A}$ such that $\bigcup_{i \in I} t(A_i) = t(A)$ and $h(A_i) = h(A)$, for each i.

We say that a directed hypergraph $\mathcal{G} = \langle V, \mathcal{A}, t, h \rangle$ is *cyclic* if there are hyperarcs $A_0, A_1, \dots, A_n$ in $\mathcal{A}$ such that $h(A_0) \subseteq t(A_1), \dots, h(A_{n-1}) \subseteq t(A_n)$ and such that $h(A_n) \in t(A_0)$. It is *acyclic* otherwise. Let a graph $\mathcal{G} = \langle V, \mathcal{A}, t, h \rangle$ be given and let $V_0 \subset V$ be some vertices. The closure of V_0 under $\mathcal{A}$ is the least $W \supseteq V_0$ such that if $V_1 \subseteq W$ and $t(A) = V_1$ then $h(A) \subseteq W$.

A *pointed* directed hypergraph is a tuple $\mathcal{G} = \langle V, F, \mathcal{A}, t, h \rangle$ where $\langle V, \mathcal{A}, t, h \rangle$ is a directed hypergraph and $F \subset V$ is a set of designated vertices such that

(i) F is closed under taking heads, that is, if $F_0 \subseteq F$ and $t(A) = F_0$, then $h(F_0) \in F$.

(ii) $\mathcal{G}$ restricted to F is acyclic.[57]

We say that F is the *actual world* of $\mathcal{G}$. Note that we do not require that $\mathcal{G}$ is acyclic: we only require that $\mathcal{G}$ is acyclic on the actual world. From now onwards, we refer to acyclic, amalgamating, chained, pointed, directed hypergraphs simply as “hypergraphs”.

We will interpret PLSFG over hypergraphs. More precisely, let the sequents of PLSFG be formed from the set of atomic letters $\mathbb{P} = \{p_0, p_1, \dots\}$. A model for PLSFG is a pair $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ where $\mathcal{G} = \langle V_{\mathcal{G}}, F_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}, t_{\mathcal{G}}, h_{\mathcal{G}} \rangle$ is a hypergraph and $[\![\ \ ]\!] \colon \mathbb{P} \to F_{\mathcal{G}}$ is function from the atomic letters into F. If Γ is a collection of atomic letters then $[\![\Gamma]\!]$ is $\bigcup_{\gamma \in \Gamma} \{[\![\gamma]\!]\}$. We call such a $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ a *graph-model*.

Truth and consequence are defined as follows.

- $\mathcal{G} \models \Gamma < \phi$ iff there is $A \in \mathcal{A}$ such that $t(A) \subseteq [\![\Gamma]\!]$, and $h(A) = [\![\phi]\!]$.
- Let S be a set of sequents and let α be a sequent. We say that α is a consequence of S ($S \models \alpha$) if for all graph-models $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ if each $\beta \in S$ is true in $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ then α is true in $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$.

It is straightforward to establish the following.

Proposition B.1. PLSFG *is sound with respect to graph-models.*

Proof: Obvious. □

Proposition B.2. PLSFG *is complete with respect to graph-models.*

Proof: Let S be a collection of sequents and α a sequent such that $S \nvdash \alpha$. We define a model $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ as follows. We let $V = \{p \colon p \text{ is an atom }\}$. We put $\mathcal{A} = \{\Gamma < \phi \colon S \vdash \Gamma < \phi\}$. We put $t(\Gamma < \phi) = \Gamma$ and $h(\Gamma < \phi) = \{\phi\}$. We put $[\![\phi]\!] = \phi$. Let $\mathcal{G} = \langle V, F, \mathcal{A}, h, t \rangle$. It is straightforward to check that $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ is a model of S in which α is not true. □

[56] This ensures that Cut and Chaining are taken care of in a simple way.

[57] That is, the subgraph generated by $\mathcal{A}$ over F is acyclic.

When we deal with iterated ground cardinality problems arise. We will assume that the set of $\mathbb{P}$ of propositional atoms is strongly inaccessible. Say that $\mathbb{P}$ has cardinality κ. The sentences of PLISFG over $\mathbb{P}$ are generated as follows. Whenever Γ is a set of sentences of cardinality $< \kappa$ and ϕ is a sentence, then $\Gamma \Rightarrow \phi$ and $\Gamma < \phi$ are sentences.

To give a semantics for iterated ground we introduce the following notion. A *graph with operators* is a tuple $\mathcal{G} = \langle V, F, \mathcal{A}, h, t, <, \Rightarrow \rangle$ where $\langle V, F, \mathcal{A}, h, t \rangle$ is a hypergraph satisfying the following constraints:

(i) V is a strongly inaccessible cardinal

(ii) $|t(A)| < |V|$ for all $A \in \mathcal{A}$

$<$ and $\Rightarrow$ are one-one functions $V^{<|V|} \to V$ such that

(iii) if A is an arc then there is $B \in \mathcal{A}$ such that $t(B) = \emptyset$ and $h(B) = t(A) \Rightarrow h(A)$;

(iv) if $X \Rightarrow v \in F$ then there is $A \in \mathcal{A}$ such that $t(A) = X$ and $h(A) = v$;

(v) if $t(A) \subseteq F$ then $t(A) < h(a) \in F$ and there is B such that $t(B) = t(A) \cup \{t(A) \Rightarrow h(A)\}$ with $h(B) = t(A) < h(A)$;

(vi) if $X < v \in F$ then $X \subseteq F$ and there is A such that $t(A) = X$ and $h(A) = v$.

Remark B.3. The reason for requiring V to have strongly inaccessible cardinality is that it is very natural to assume the (Distinctness Principle):

(Distinctness Principle) If $V_0 \neq V_1$ or $v_0 \neq v_1$ then $(V_0 \Rightarrow v_0) \neq (V_1 \Rightarrow v_1)$.

(The truth that V_0 non-factively ground v_0 and the truth that V_1 non-factively ground v_1 seem distinct.)

If there are κ-many vertices a graph-model might have as many as 2^κ-many arcs. The (Distinctness Principle) would then ensure that there are 2^κ many vertices after all. We therefore impose some constraints on the graphs.

A model for PLISFG is a pair $\langle \mathcal{G}, [\![\;\;]\!] \rangle$ such that $\mathcal{G}$ is a graph with operators and $[\![\;\;]\!] : \mathbb{P} \to V_{\mathcal{G}}$ is a function from the propositional atoms into the vertices of $\mathcal{G}$. We here demand that V has greater cardinality than the set of sentence letters.

We extend the interpretation function from the atomic letters to arbitrary formulae in the obvious way.

- $[\![\Gamma \Rightarrow \phi]\!] = [\![\Gamma]\!] \Rightarrow [\![\phi]\!]$
- $[\![\Gamma < \phi]\!] = [\![\Gamma]\!] < [\![\phi]\!]$

Hypothetical arguments are dealt with as follows: If $\Gamma \Vdash_p \phi$ is a plain hypothetical argument we say that $\mathcal{G}$ *validates* $\Gamma \Vdash_p \phi$ if whenever $[\![\Gamma]\!] \subseteq F$, $[\![\phi]\!] \subseteq F$. If $\Gamma \Vdash_e \phi$ is a hypothetical explanatory argument we say that $\mathcal{G}$ *validates* $\Gamma \Vdash_e \phi$ if there is an arc $A \in \mathcal{A}_{\mathcal{G}}$ with $[\![\Gamma]\!] = t(A)$ and $[\![\phi]\!] = h(A)$.

The following soundness theorem is easily established.

Proposition B.4. *Let explanatory and plain arguments $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ be given.*

(i) If there is a plain argument from Γ to ϕ in PLISFG *over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ then for all models $\mathcal{G}$ validating $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ we have that if $[\![\Gamma]\!] \subseteq F$ then $[\![\phi]\!] \in F$.*

(ii) If there is an explanatory argument from Γ to ϕ in PLISFG *over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ then for all models $\mathcal{G}$ validating $\langle \mathfrak{E}_s, \mathfrak{E}_p \rangle$ there is an arc A such that $t(A) = [\![\Gamma]\!]$ and $h(A) = [\![\phi]\!]$.*

Proof: We prove the claims simultaneously by induction on the complexity of the arguments witnessing that there is a plain (explanatory) argument from Γ to ϕ. □

One can also establish the following completeness theorem.

Theorem B.5. *Let $\mathfrak{E}_e, \mathfrak{E}_p$ be some explanatory and plain arguments. If α is a statement that is not derivable from $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ then there is a graph with operators $\mathcal{G}$ and an evaluation $[\![\ \ ]\!]$ such that α is not true over $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$.*

Proof: By a straightforward extension of the proof of Proposition B.2. □

To establish that PLISFG is a conservative extension of PLSFG we establish the following proposition. (As stated we require a proper class of inaccessible cardinals.)

Proposition B.6. *Let $\mathcal{G} = \langle V, F, \mathcal{A}, t, h \rangle$ be a hypergraph. There is a graph with operators $\mathcal{G}^+ = \langle V^+, F^+, \mathcal{A}^+, \Rightarrow, <, t^+, h^+ \rangle$ such that*

(i) $V \subseteq V^+, \mathcal{A} \subseteq \mathcal{A}^+, h \subseteq h^+, t \subseteq t^+$

(ii) $\mathcal{A}^+$ restricted to V is $\mathcal{A}$;

(iii) h^+, t^+ restricted to V is h, t.

Sketch: Let κ be the cardinality of V. Let λ be the least strongly inaccessible cardinal $> \kappa$. We extend $\mathcal{G}$ to a graph with operators in λ-many stages $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_\gamma, \ldots$ in the obvious way. □

Theorem B.7. *Let S be a collection of sequents in the language of* PLSFG *and let $\Gamma < \phi$ be a sequent in the language of* PLSFG*. If $S \vdash_{\text{PLSFG}} \Gamma < \phi$ then $S \vdash_{\text{PLISFG}} \Gamma < \phi$.*

Proof: Let $\langle \mathcal{G}, [\![\ \ ]\!] \rangle$ be a model witnessing that $S \vdash \Gamma < \phi$. Say that $V_{\mathcal{G}}$ has cardinality κ. We can extend $\mathcal{G}$ to a graph with operators $\mathcal{G}^+ = \langle V^+, F^+, \mathcal{A}^+, \Rightarrow, <, t^+, h^+ \rangle$ satisfying the conditions in Proposition B.6. The interpretation $[\![\ \ ]\!]$ is extended to an interpretation $[\![\ \ ]\!]^+$ of the language of PLISFG in the obvious way. The restriction of $\mathcal{A}^+$ to V is $\mathcal{A}$. It follows that $\Gamma < \phi$ is not true in $\langle \mathcal{G}^+, [\![\ \ ]\!]^+ \rangle$. □